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Pitch Angle Control of Unmanned Air Vehicle with Uncertain System Parameters

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Abstract In this paper a new algorithm of trajectory tracking based on closest radius solution of the interval equations system is proposed. The design procedure is given and applied to the pitch angle control of unmanned testing rocket with uncertain parameters. The proposed algorithm gives a framework to design a control for a wide range of different linear time-invariant processes with uncertain parameters and can be implemented also in the case of non-convex problems. The algorithm gives the analytical way of finding the nearly optimal solution of model reference trajectory tracking in the case of general time-invariant systems with uncertain parameters and can be used when optimization method fails due to the complexity of the problem.

Key words control • interval model • linear interval solution

1 Introduction

The problem of pitch angle control of the testing rocket (Japan Aerospace Exploration Agency, Institute of Space and Astronautical) becomes much more demanding when taking into account the changes of aerodynamic parameters. The aerodynamic parameters which are not know exactly can also vary according to the actual weather conditions and the speed of the rocket. The change of these parameters causes a serious problem by the design of control which should be robust in the whole range of parameter changes and should give an appropriate control performance.

The problem of robust control design for the linear time-invariant systems with uncertain parameters has received considerable attention in recent years. Various design techniques have been presented. The methods where exact pole assignment is introduced are shown in Harvey and Stein [10], Sebakhy [20] and Šiljak [21].

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Furuta and Kim [7], Kim and Lee [12] and Lee and Lee [13] proposed the feedback controller which minimize a cost function subject to the requirements that the closed-loop poles lies with in a specified region. In Haddad and Bernstein [9] the modified Lyapunov function is solved to minimize an auxiliary performance index which guaranties the upper bound of the quadratic cost function and guaranties the specified region of the closed-loop poles. The multi-constraints optimal regional pole placement problem is given in Wu and Lee [23] where the closed-loop poles are placed in a pre-specified region. An approach which is based on the interval polynomial theory to design a robust pole-placement controller is discussed by Soh and Betz [22]. Evans [6] proposed the optimization to search an optimal compensator to robustly stabilize the interval system. A numerical technique to design a robust stabilizing controller for uncertain interval plants is given in [8]. Rotstein [19] proposed a mathematical programming method to design a robust controller. The design using Kharitonov theorem is given by Barmish and Tempo [1], Bernstein [2] and Chapellat and Bhattacharyya [4]. The stability of polynomials under coefficient perturbations is studied by Bialas and Garloff [3]. The technique where the robust design problem is transformed to nonlinear constrained optimization problem is given by Chen [5].

In our approach the problem of pole region assignment is transformed to the problem of solving the set of interval algebraic equations [18]. The main objective is to find the interval solution the sense of the closest radius [17]. This means that the smallest polytope which contains all possible solutions has to be defined. In other words, the set of all possible solutions should be imbedded into the minimal polytope in problem space domain. The desired dynamic of the closed-loop (the reference model trajectory) is given by dominant pole.

The paper is organized in the following way: in Section 2 the dynamics of the unmanned testing rocket with uncertain parameters is shown, Section 3 describes the proposed control of the rocket with uncertain physical parameters, in Section 4 closest radius solution design for the winged body with uncertain physical parameters is shown, Section 5 shows the simulation results and at the end the conclusions are given.

2 Pitch Angle Dynamics of Unmanned Testing Rocket with Uncertain Parameters

The problem of robust control design for the linear time-invariant systems with uncertain parameters has received considerable attention in recent years. The pitch angle dynamics of the testing rocket belongs to the group of systems with uncertain parameter because of the changing aerodynamics parameters. In this study only the longitudinal motion of the rocket will be investigated, i.e., only the transfer function $\frac{\Theta(s)}{\delta(s)}$, where Θ is pitch angle and δ is control surface deflection angle is given. The diagram of forces and moments is shown in Figure 1, where x and z represent the coordinates of the system, V is the airspeed, u and w are the airspeed components in x and z coordinates, Θ is the pitch angle, γ is the angle of attack, Q is the angular velocity, F_{g} stands for the gravity force, F_{L} is the lift force, F_{D} the drag force, F_{x} is the air pressure force in the direction of x coordinate, F_{z} is the air pressure force in the direction of z coordinate, M is the momentum caused by lift and drag, F_{c} stands for the control force, cp is the center of pressure, cg is the center of gravity, l_{cp} is the handle of pressure center and l_{c} is the handle of control force. The nonlinear

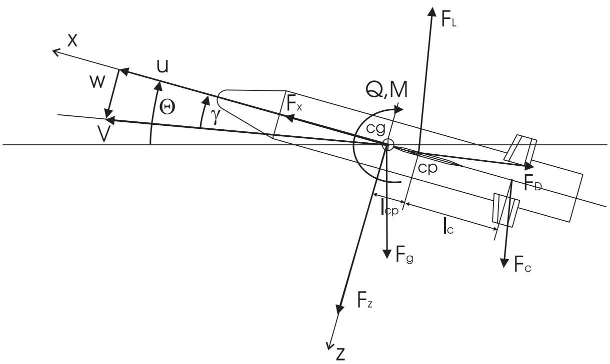


Figure 1 The diagram of forces and torques.

model of the rocket dynamics is described in [14]. To design the pitch angle control, the model is linearized in the operating point $u = 150$ m/s, $w = 0$, $\gamma = 0$, $Q = 0$ and $\Theta = 0$. The following transfer function which describes the dynamics between the pitch angle velocity Ω and the angle of the deflection fins δ is obtained:

$$G(s) = \frac{\Omega(s)}{\delta(s)} = \frac{(\tilde{\nu}_1(\boldsymbol{q})s + \tilde{\nu}_0(\boldsymbol{q}))}{\left(s^2 + \tilde{\mu}_1(\boldsymbol{q})s + \tilde{\mu}_0(\boldsymbol{q})\right)} \tag{1}$$

with $\boldsymbol{q}$ as the parameter vector which consists of $\boldsymbol{q} = \left[l;\, m;\, S;\, I_y;\, l_c;\, l_{cp};\, C_{M\alpha};\, \tilde{C}_{L\alpha};\, \tilde{C}_{L\delta}\right]$ and where $\tilde{\nu}_0(\boldsymbol{q})$, $\tilde{\nu}_1(\boldsymbol{q})$, $\tilde{\mu}_0(\boldsymbol{q})$ and $\tilde{\mu}_1(\boldsymbol{q})$ stand for:

$$\tilde{\nu}_0(\boldsymbol{q}) = n_e \frac{q^2 S^2 \tilde{C}_{L\delta}}{m u I_y}\left(\tilde{C}_{L\alpha} l_c + C_{M\alpha} l\right) \tag{2}$$

$$\tilde{\nu}_1(\boldsymbol{q}) = n_e \frac{q S \tilde{C}_{L\alpha} l_c}{I_y} \tag{3}$$

$$\tilde{\mu}_0(\boldsymbol{q}) = -\frac{q S C_{M\alpha} l}{I_y} \tag{4}$$

$$\tilde{\mu}_1(\boldsymbol{q}) = \frac{q S \tilde{C}_{L\alpha}}{m u} + \frac{q S C_{M\alpha} l_{cp}}{I_y u} \tag{5}$$

The coefficients of the parameters vector $\boldsymbol{q}$ are equal $l = 3.386$ m, $m = 260$ kg, $S = 0.12566$ m^2, $I_y = 253.1$ kgm^2, $l_c = -0.8285$ m, $l_{cp} = -0.0921$ m, $C_{M\alpha} = -0.6393$, and the lift coefficients are given as independently varying interval parameters $\tilde{C}_{L\alpha} =$

[14.1000; 32.9000], $\tilde{C}_{L\delta} = [1.8118;\ 4.2276]$ and the constant $n_e = 2\sqrt{2}$. This implies the uncertain parameters of the transfer function from Eq. 1 given as

$$\tilde{\nu}_0(\boldsymbol{q}) = [-37.4230;\ -7.7144]$$

$$\tilde{\nu}_1(\boldsymbol{q}) = [-31.1822;\ -14.8214]$$

$$\tilde{\mu}_1(\boldsymbol{q}) = [0.6482;\ 1.3278]$$

$$\tilde{\mu}_0(\boldsymbol{q}) = 15.1133$$

This means that $\nu_0 = -22.5687$, $\Delta\nu_0 = 14.8543$, $\nu_1 = -23.0018$, $\Delta\nu_1 = 8.1804$, $\mu_1 = 0.9880$ and $\Delta\mu_1 = 0.3398$.

3 Control of the Rocket with Uncertain Physical Parameters

The goal of the attitude control is to control the pitch angle Θ by the manipulated variable δ using the classical compensator with the following control law:

$$\delta(s) = K_2\left(\Theta_r(s) - \Theta(s)\right) - K_1\Omega(s) \tag{6}$$

where Θ_r stands for the reference signal, with $\Omega(s) = s\Theta(s)$ and the compensator parameters K_1 and K_2.

Assuming the proposed controller structure and the controlled process with uncertain physical parameters the closed-loop characteristic polynomials are obtained in the parametric form as

$$p\left(s, \boldsymbol{q}, \boldsymbol{k}\right) = \sum_{i=0}^{n} p_i\left(\boldsymbol{q}, \boldsymbol{k}\right) s^i \tag{7}$$

where $\boldsymbol{q}$ is the uncertain parameter vector and the vector $\boldsymbol{k}$ of order m contains the free design parameters. The uncertain parameter vector consists of interval parameters q_i which are described by its lower and upper bounds q_i^- and q_i^+. When the vectors $\boldsymbol{k}$ and $\boldsymbol{q}$ are unspecified this is called an uncertain interval polynomial. If the coefficients of the vector $\boldsymbol{q}$ belongs to the operating domain Q, i.e., $\boldsymbol{q} \in Q$ and if the controller equals $\boldsymbol{k} = \boldsymbol{k}^o$, then polynomial $p\left(s, \boldsymbol{q}, \boldsymbol{k}\right)$ generates a polynomial family $P\left(s, Q, \boldsymbol{k}^o\right) = \left\{p\left(s, \boldsymbol{q}, \boldsymbol{k}^o\right) \mid \boldsymbol{q} \in Q\right\}$.

First of all we have to find the admissible robust stable set of solutions, i.e., the set of free parameters which stabilize the system. The admissible robust stable set of solutions contains all $\boldsymbol{k} = \boldsymbol{k}^o$ such that the polynomial family $P\left(s, Q, \boldsymbol{k}^o\right)$ is stable.

Let us assume an uncertain interval polynomial with uncertain physical parameters $\boldsymbol{q}$ and unknown free parameters $\boldsymbol{k}$ described as in Eq. 7. The admissible stable set of solutions K contains all $\boldsymbol{k} = \boldsymbol{k}^o$ such that the polynomial family $P\left(s, Q, \boldsymbol{k}^o\right)$ is stable, i.e., $K = \{\boldsymbol{k} \mid\ P(s, Q, \boldsymbol{k}) = (s - s_1)(s - s_2)\ldots(s - s_n),\ Re(s_i) < 0,\ i = 1, \ldots, n\}$.

To find the admissible set of stable solutions for linear time-invariant system control the Kharitonov criterion [11] is used. The closed-loop transfer function taking into account the control law from Eq. 6 is given as:

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{k_2\left(\tilde{\nu}_1 s + \tilde{\nu}_0\right)}{s^3 + \left(\tilde{\mu}_1 + k_1\tilde{\nu}_1\right)s^2 + \left(\tilde{\mu}_0 + k_1\tilde{\nu}_0 + k_2\tilde{\nu}_1\right)s + k_2\tilde{\nu}_0}$$

and the transfer function which defines the control signal becomes

$$\frac{\Delta(s)}{\Theta_r(s)} = \frac{k_2 s\left(s^2 + \tilde{\mu}_1 s + \tilde{\mu}_0\right)}{s^3 + \left(\tilde{\mu}_1 + k_1\tilde{\nu}_1\right)s^2 + \left(\tilde{\mu}_0 + k_1\tilde{\nu}_0 + k_2\tilde{\nu}_1\right)s + k_2\tilde{\nu}_0}$$

The interval characteristic polynomial of pitch angle control system is given in the following form

$$P(s) = s^3 + \left(\tilde{\mu}_1 + k_1\tilde{\nu}_1\right)s^2 + \left(\tilde{\mu}_0 + k_1\tilde{\nu}_0 + k_2\tilde{\nu}_1\right)s^1 + k_2\tilde{\nu}_0 \tag{8}$$

with k_1 and k_2 defined as free design parameters. Defining all four Kharitonov polynomial and using Routh–Hurwitz criterion the admissible stable set of solutions is obtained as follows:

$$K^s = \left\{\boldsymbol{k} = \left[k_1; k_2\right] \mid \tilde{\mu}_1 + k_1\tilde{\nu}_0 < 0, \tilde{\mu}_0 + k_1\tilde{\nu}_0 < 0, k_2\tilde{b}_0 > 0\right\} \tag{9}$$

In the case of chosen parameters the admissible stable set becomes

$$K^s = \left\{\boldsymbol{k} = \left[k_1;\ k_2\right] \mid \ k_1 < 0.0355,\ k_2 < 0\right\} \tag{10}$$

The desired closed-loop polynomial are calculated according to the control law from Eq. 6 as follows:

$$D(s) = d_3 s^3 + d_2 s^2 + d_1 s^1 + d_0 \tag{11}$$

$$d_3 = 1 \tag{12}$$

$$d_2 = \tilde{\mu}_1(\boldsymbol{q}) + \tilde{\nu}_1(\boldsymbol{q})k_1 \tag{13}$$

$$d_1 = \tilde{\nu}_0(\boldsymbol{q})k_1 + \tilde{\nu}_1(\boldsymbol{q})k_2 + \tilde{\mu}_0(\boldsymbol{q}) \tag{14}$$

$$d_0 = \tilde{\nu}_0(\boldsymbol{q})k_2 \tag{15}$$

The desired closed-loop poles are obtained to place one of the poles in the closed loop zero which is equal to $-\nu_0/\nu_1$, second pole is defined to lie fare on the left side and is equal to -20 and the third is the dominant pole and is equal to -2. The desired closed-loop coefficients are than equal to $d_2 = 22.9812,\ d_1 = 61.5857,\ d_0 = 39.2468$.

To find the admissible set of solutions to be as close as possible to the desired poles for the whole set of interval parameters the next interval matrix equation has to be solved

$$\begin{bmatrix} \tilde{\nu}_1(\boldsymbol{q}) & 0 \\ \tilde{\nu}_0(\boldsymbol{q}) & \tilde{\nu}_1(\boldsymbol{q}) \\ 0 & \tilde{\nu}_0(\boldsymbol{q}) \end{bmatrix} \boldsymbol{k} = \begin{bmatrix} d_2 - \tilde{\mu}_1(\boldsymbol{q}) \\ d_1 - \tilde{\mu}_0(\boldsymbol{q}) \\ d_0 \end{bmatrix} \tag{16}$$

4 Closest Radius Solution of Interval Matrix Equation

The closest radius solution of interval matrix equation [15] means to find the free design parameters $\boldsymbol{k} = \boldsymbol{k}^o$ to place the closed-loop poles of the whole family of processes as close as possible to the prescribed closed-loop poles.

Let us define the desired characteristic closed-loop polynomial in the form

$$D(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0 \tag{17}$$

The problem of closest radius solution is to find the set of all admissible solutions K of the following interval equalities

$$p_i\left(\boldsymbol{q}, \boldsymbol{k}\right) = d_i, \quad i = 0, \ldots, n-1 \tag{18}$$

The controller structure is usually assumed which leads to the closed-loop characteristic polynomial in the affine parametric form as

$$d_i = \boldsymbol{a}_i^T(\boldsymbol{q})\,\boldsymbol{k} + b_i(\boldsymbol{q}), \quad i = 0, \ldots, n-1 \tag{19}$$

The closest radius solution in the case of time-invariant systems with uncertain parameters is considered as the problem of the best interval solution for interval system of linear algebraic equations in Eq. 19 which can be written in interval matrix form as

$$\tilde{\boldsymbol{A}}\boldsymbol{k} = \tilde{\boldsymbol{b}} \tag{20}$$

with $\boldsymbol{k} \in \mathbb{R}^m$, the interval matrix $\tilde{\boldsymbol{A}} \in \mathbb{R}^{n\times m}$ and the interval vector $\tilde{\boldsymbol{b}} \in \mathbb{R}^n$. Equation 20 can be also written in the form

$$\left(\boldsymbol{A} \pm \Delta\boldsymbol{A}\right)\boldsymbol{k} = \boldsymbol{b} \pm \Delta\boldsymbol{b} \tag{21}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{A} \pm \Delta\boldsymbol{A}$ and $\tilde{\boldsymbol{b}} = \boldsymbol{b} \pm \Delta\boldsymbol{b}$. The elements of matrix $\tilde{\boldsymbol{A}}$ and the elements of vector $\tilde{\boldsymbol{b}}$ are described as

$$\tilde{a}_{ij} = \left[a_{ij} - \Delta a_{ij};\; a_{ij} + \Delta a_{ij}\right] \tag{22}$$

$$\tilde{b}_i = \left[b_i - \Delta b_i;\; b_i + \Delta b_i\right] \quad , i = 1, \ldots, n, \;\; j = 1, \ldots, m \tag{23}$$

When the element of perturbed matrix $\tilde{a}_{i,j} = 0$ than it means that $a_{i,j} = 0$ and $\Delta a_{i,j} = 0$.

The set of admissible solution for the interval equation from Eq. 20 which is defined in the sense of closest radius solution, i.e., the solution within minimal radius is defined by the following lemma.

Lemma 1 The set of all admissible solutions of the interval matrix inequality is a polytope ([16]):

$$K = \{\boldsymbol{k} \mid \mid \boldsymbol{A}\boldsymbol{k} - \boldsymbol{b} \mid_\infty \leq \epsilon \mid \boldsymbol{k} \mid_1 + \delta\} \tag{24}$$

with $\mid \boldsymbol{A}\boldsymbol{k} - \boldsymbol{b} \mid_\infty = \max_{i,j} \mid a_{i,j}k_j - b_i \mid$, $\mid \boldsymbol{k} \mid_1 = \sum_{i=1}^m \mid k_i \mid$ and $\left\|\Delta\boldsymbol{A}\right\|_\infty \leq \epsilon$ and $\mid \Delta\boldsymbol{b} \mid_\infty \leq \delta$.

The analytical way to find the admissible set of solutions is to solve each row of the matrix inequality from Eq. 24 separately.

Lemma 2 The interval matrix inequality can be rewritten in n interval inequalities as follows:

$$\left|\boldsymbol{a}_i^T\boldsymbol{k} - b_i\right| \leq \epsilon_i \left|\boldsymbol{e}_i^T\boldsymbol{k}\right| + \delta_i, \quad i = 1, \ldots, n \tag{25}$$

with $|\Delta\boldsymbol{a}_i|_\infty \leq \epsilon_i$ and $|\Delta b_i| \leq \delta_i$ and $\boldsymbol{a}_i$ stands for ith row of matrix $\boldsymbol{A}$ $\left(\boldsymbol{a}_i^T = \left[a_{i,1}; \ldots; a_{i,j}; \ldots; a_{i,m}\right]\right)$. The vector $\boldsymbol{e}_i^T \in \mathbb{R}^{m\times 1}$ is defined as follows:

$$\boldsymbol{e}_i^T = \left[\ f(a_{i,1}) \cdots f(a_{i,j}) \cdots f(a_{i,m})\ \right] \tag{26}$$

where the diagonal elements are defined as $f(a_{i,j}) = 1$ if $a_{i,j} \neq 0$ and $f(a_{i,j}) = 0$ if $a_{i,j} = 0$ for $i = 1, ..., n,\ \ j = 1, ..., m$.

Proof The ith row in interval matrix equality (21) is written as

$$\left(\boldsymbol{a}_i^T \pm \Delta\boldsymbol{a}_i\right)\boldsymbol{k} = b_i \pm \Delta b_i \tag{27}$$

The obtain the minimal radius of the interval matrix solutions the infinity norm is applied to Eq. 27 that is rewritten in a form where nominal terms are on left side and than yields

$$\left|\boldsymbol{a}_i^T\boldsymbol{k} - b_i\right| \ = \ |\Delta\boldsymbol{a}_i\boldsymbol{k} + \Delta b_i| \tag{28}$$

Introducing the Minkowski's inequality than yields

$$\left|\boldsymbol{a}_i^T\boldsymbol{k} - b_i\right| \ \leq |\ \Delta\boldsymbol{a}_i\boldsymbol{k}\ | + |\ \Delta b_i\ | \tag{29}$$

and by introducing basic relation between vector and matrix norms it follows

$$\left|\boldsymbol{a}_i^T\boldsymbol{k} - b_i\right| \ \leq |\ \Delta\boldsymbol{a}_i\ ||\ \boldsymbol{e}_i^T\boldsymbol{k}\ | + |\ \Delta b_i\ | \tag{30}$$

if we define $|\Delta\boldsymbol{a}_i|_\infty \leq \epsilon_i$ and $|\Delta b_i| \leq \delta_i$ we get Eq. 25. □

The solution of Eq. 25 on free parameters $\boldsymbol{k}$ leads to the optimal region assignment in the smallest radius sense. The solution is in general a non-convex polytope. The solution in the form of vertices is very difficult to find. This problem is known as the enumeration problem. In our approach the solution is find by using the triangular inequality [11] which results in more conservative solution in the form of convex polytope which is a subset of the solution defined in Eq. 25.

A simple and effective approach to solve the non-convex problem given in Eq. 25 will be discussed next. Using the triangular inequality $|\ a\ | - |\ b\ | \leq |\ a - b\ |$ on Eq. 25 yields

$$\left|\left(\boldsymbol{a}_i^T - \epsilon_i\boldsymbol{e}_i^T\right)\boldsymbol{k} - b_i\right| \leq \delta_i, \quad i = 1, \ldots, n \tag{31}$$

Let us introduce a weight parameters $w_p = 1$ and $w_n = -1$. Equation 31 is therefore described in the following form

$$w_p\left(\boldsymbol{a}_i^T - \epsilon_i\boldsymbol{e}_i^T\right)\boldsymbol{k} \leq \delta_i + w_p b_i \tag{32}$$

$$w_n\left(\boldsymbol{a}_i^T - \epsilon_i\boldsymbol{E}_i^T\right)\boldsymbol{k} \leq \delta_i + w_n b_i, \quad i = 1, \ldots, n \tag{33}$$

Each equation from Eq. 25 results in two m-dimensional hyperplane which are parallel to each other. Equation 33 can be rewritten in a more compact form as

$$\boldsymbol{a}_{e_{(i-1)\cdot 2+\rho}}^T\boldsymbol{k} \leq \ b_{e_{(i-1)\cdot 2+\rho}}, \quad i = 1, \ldots, n \ \ \rho = 1, 2 \tag{34}$$

with

$$\boldsymbol{a}^T_{e_{(i-1)\cdot 2+1}} = w_p \left(\boldsymbol{a}_i^T - \epsilon_i \boldsymbol{e}_i^T\right), \; b_{e_{(i-1)\cdot 2+1}} = \delta_i + w_p b_i$$

$$\boldsymbol{a}^T_{e_{(i-1)\cdot 2+2}} = w_n \left(\boldsymbol{a}_i^T - \epsilon_i \boldsymbol{e}_i^T\right), \; b_{e_{(i-1)\cdot 2+2}} = \delta_i + w_n b_i$$

for $i = 1, \ldots, n$.

Equation 34 can be in matrix form written as

$$\boldsymbol{A}_e \boldsymbol{k} \leq b_e \tag{35}$$

with $\boldsymbol{A}_e \in \mathbb{R}^{n\cdot 2\times m}$ and $\boldsymbol{k} \in \mathbb{R}^m$.

Equation 35 defines the polytope which describes the admissible regions of all free parameters. To find the vertices of this polytope we have to find the solutions of all possible combination of m equations in the whole set of $n \cdot 2$ equations. The set of solution has $\binom{n\cdot 2}{m}$ elements and is given as follows

$$\begin{aligned} K^i_{min} = \Big\{ \boldsymbol{k} \mid \; & \boldsymbol{a}^T_{e_{i_1}} \boldsymbol{k} = b_{e_{i_1}}, \ldots, \boldsymbol{a}^T_{e_{i_m}} \boldsymbol{k} = b_{e_{i_m}}, \\ & i_1 = 1, \ldots, n\cdot 2 - m + 1, \ldots, i_m = i_1 + m - 1, \ldots, n \cdot 2 \Big\} \end{aligned}$$

The set K^i_{min} consists of all possible solutions but only those which satisfy also the condition $\boldsymbol{A}_e \boldsymbol{k} \leq b_e$ are extremal solutions known as vertices of the polytope. The set of vertices which defines the set of admissible solutions is a convex set and is described as

$$K^v_{min} = \left\{ \boldsymbol{k} \mid \; \boldsymbol{k} \in K^i_{min}, \; \boldsymbol{A}_e \boldsymbol{k} \leq b_e \right\} \tag{36}$$

The solution using triangular inequality is a very conservative and the solution does not always exist. When the minimal set K^i_{min} does not exist, then the solution should be find be solving the set of equations in Eq. 25 which can be now written as

$$\begin{aligned} & \left(w_{i,m+1} a_{i,1} - \epsilon_i w_{i,1} f(a_{i,1})\right) k_1 + \cdots + \left(w_{i,m+1} a_{i,m} - \epsilon_i w_{i,m} f(a_{i,m})\right) k_m \\ & \quad \leq \; w_{i,m+1} b_i - \delta_i i = 1, \ldots, n \end{aligned}$$

with the set of vectors $\boldsymbol{w}_i \in \mathbb{R}^{m+1}, \; w_{i,j} = \pm 1, \; i = 1, \ldots, n$ and $j = 1, \ldots, m+1$. The dimension of vector $\boldsymbol{w}_i$ is $m+1$, i.e., this means that we have 2^{m+1} different variation of this vector. Each row from Eq. 25 results in 2^{m+1} new inequalities. When we form the matrix $\boldsymbol{A}_e$ it belongs to $\boldsymbol{A}_e \in \mathbb{R}^{n\cdot 2^{m+1}\times m}$.

The solution is now defined as maximal set K^i_{max} defined as

$$\begin{aligned} K^i_{max} = \Big\{ \boldsymbol{k} \mid \; & \boldsymbol{a}^T_{e_{i_1}} \boldsymbol{k} = b_{e_{i_1}}, \ldots, \boldsymbol{a}^T_{e_{i_m}} \boldsymbol{k} = b_{e_{i_m}}, \\ & i_1 = 1, \ldots, n \cdot 2^{m+1} - m + 1, \ldots, \\ & i_m = i_1 + m - 1, \ldots, n \cdot 2^{m+1} \Big\} \end{aligned}$$

Taking into account Eq. 35 the maximal set of vertices K^v_{max} is obtained

$$K^v_{max} = \left\{ \boldsymbol{k} \mid \; \boldsymbol{k} \in K^i_{max}, \; \boldsymbol{A}_e \boldsymbol{k} \leq b_e \right\} \tag{37}$$

To define the design parameter $\boldsymbol{k}_o$ we have to find first the set K^v_{min}. If it is empty, i.e., it does not exist than in second step we have to find the set K^v_{max}. Both sets will be further denoted as K^v. When the set K^v is defined, the exact value of the design parameter should be defined. This calculation depends on the number of elements in the set K^v. In case when $n_v = 2^m$ the first momentum is used as follows:

$$\boldsymbol{k}_o = \frac{1}{n_v} \sum_{i=1}^{n_v} \boldsymbol{k}_i \tag{38}$$

where $\boldsymbol{k}_i \in K^v,\ i = 1, \ldots, n_v$ and with n_v as a number of elements in set K^v.

When $n_v > 2^m$ then the problem is solved by assuming a minimal orthogonal convex approximation of the solution set. This solution consists only from the minimal and maximal values at each dimension $k_i,\ i = 1, \ldots, m$ where $\boldsymbol{k} \in K^v$. If we define the minimal values as $\underline{k}_i = \min_{\boldsymbol{k}\in K^v},\ i = 1, \ldots, m$ and the maximal values as $\overline{k}_i = \max_{\boldsymbol{k}\in K^v},\ i = 1, \ldots, m$ then we have to find those elements in K^v which have the minimal or maximal values in the all rest dimensions. This leads to the extremal approximative solution set defined as K^e which has 2^m elements and than again the first momentum can be used to define $\mathbf{k}_o$.

4.1 Closest Radius Solution in the Rocket Case

The closest radius solution in the rocket case is obtained in the following way. Equation 16 is transformed, by taking into account Eq. 25, into the following three inequalities:

$$|\nu_1 k_1 + \mu_1 - d_2| \leq \epsilon_1 |k_1| + \delta_1 \tag{39}$$

$$|\nu_0 k_1 + \nu_1 k_2 + \mu_0 - d_1| \leq \epsilon_2 \left(|k_1| + |k_2|\right) + \delta_2 \tag{40}$$

$$|\nu_0 k_2 - d_0| \leq \epsilon_3 |k_2| + \delta_3 \tag{41}$$

where $\epsilon_1 = \Delta\nu_1,\ \delta_1 = \Delta\mu_1,\ \epsilon_2 = \max(\Delta\nu_0, \Delta\nu_1),\ \delta_2 = \Delta\mu_0$ and $\epsilon_3 = \Delta\nu_0,\ \delta_3 = 0$. The minimal approximative solution in this case does not exist $K^v_{min} = \{\}$. This means that the set is convex and the maximal approximative solution is conformable to the exact solution of Eq. 41. The maximal approximative set is given as

$$K^v_{max} = \{\boldsymbol{k}_1,\ \boldsymbol{k}_2,\ \boldsymbol{k}_3,\ \boldsymbol{k}_4\} \tag{42}$$

where

$$\begin{aligned}
\boldsymbol{k}_1 &= [-1.5068; -1.0487] \\
\boldsymbol{k}_2 &= [-0.7162; -1.0487] \\
\boldsymbol{k}_3 &= [-0.7162; -5.0258] \\
\boldsymbol{k}_4 &= [-1.5068 - 4.2772]
\end{aligned}$$

The polytope can be presented by the minimal set of inequations which are give in matrix form as

$$\begin{bmatrix} -14.8214 & 0 \\ 7.7144 & 8.1475 \\ 0 & -37.4230 \\ -31.1822 & 0 \end{bmatrix} \boldsymbol{k} \leq \begin{bmatrix} 22.3330 \\ -46.4724 \\ 39.2468 \\ 22.3330 \end{bmatrix} \tag{43}$$

Figure 2 The admissible set of solutions.

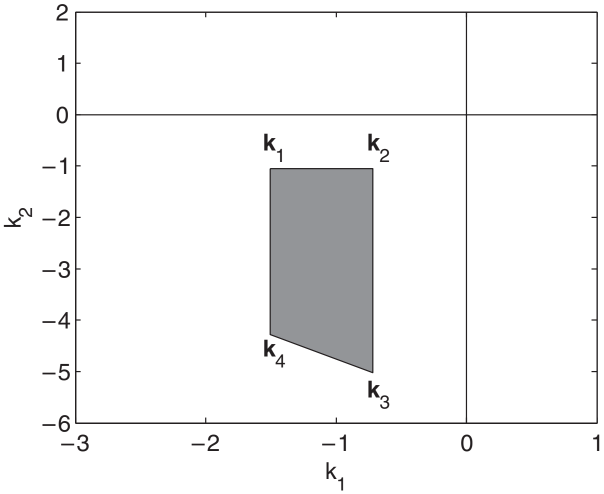
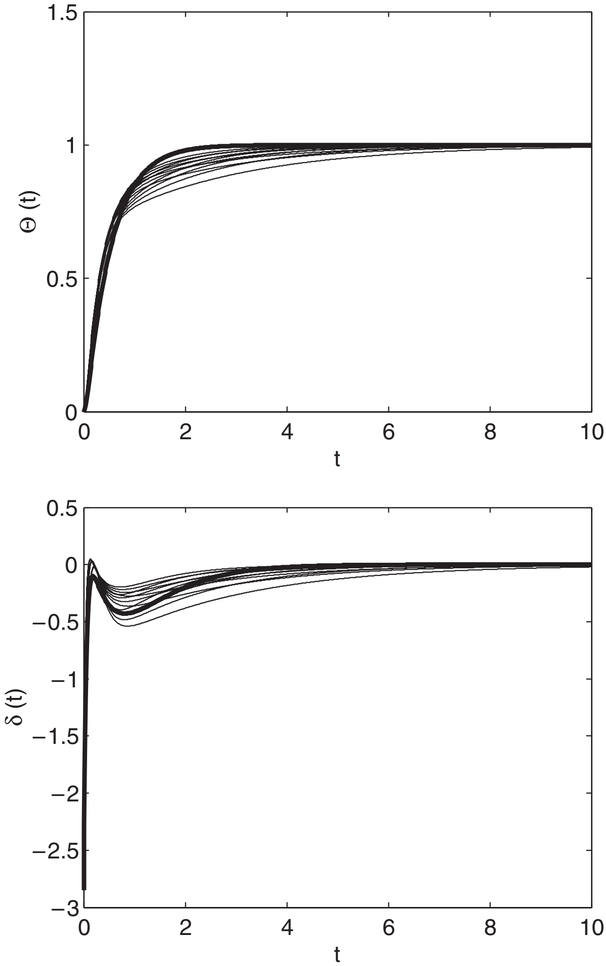


Figure 3 The whole family of time responses for free design parameter $\boldsymbol{k}_o$.



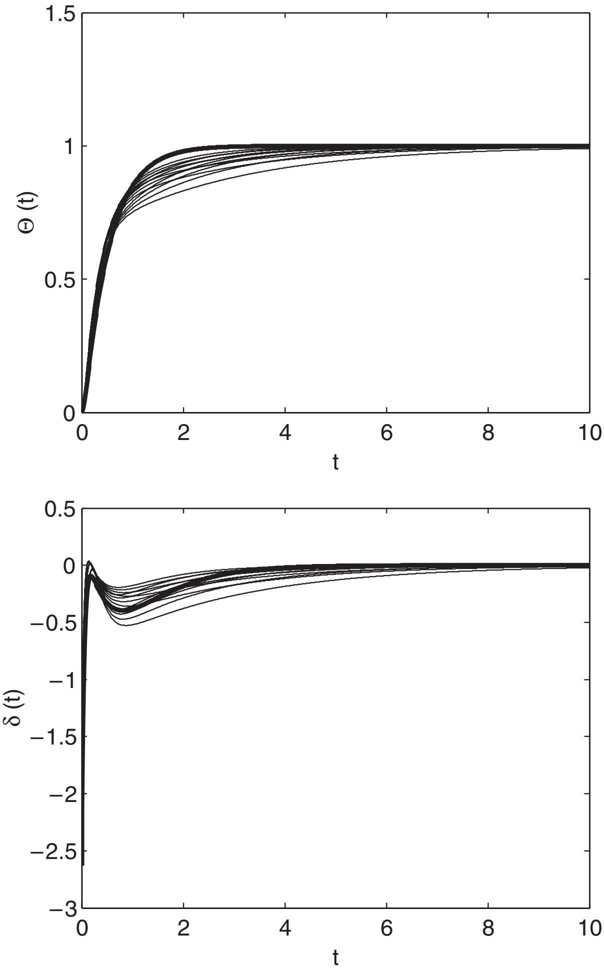
The vertices of this set and the whole admissible set are presented in Figure 2. The optimal free design parameter is taken as the first momentum of the set K^v_{max} and it is given as $\boldsymbol{k}_o = [-1.1115;\ -2.8501]$.

5 Simulation Study

In the simulation study the comparison of the proposed design method to the optimization method is shown. The optimization method is based on finding free design parameter $\boldsymbol{k}_d$ to minimize the cost function

$$J = \sum_{i=1}^{m} \| y_d(t) - y(\boldsymbol{q}_i, \boldsymbol{k}, t)\|_2^2 + \sum_{i=1}^{m} \| u(\boldsymbol{q}_i, \boldsymbol{k}, t)\|_\infty \tag{44}$$

Figure 4 The whole family of time responses for free design parameter $\boldsymbol{k}_d$.



with the constraint $\|u(\boldsymbol{q}_i, \boldsymbol{k}, t)\|_\infty \leq 3$ and y_d as the desired model reference trajectory which is defined by the desired closed-loop polynomial in Eq. 13. Figure 3 shows the whole family of different responses with constant free parameter $\boldsymbol{k}_o$ and uncertain transfer function parameters with $\boldsymbol{q}_i,\ i = 1, \ldots, 12$ where the interval parameters $\tilde{C}_{L\alpha}$ and $\tilde{C}_{L\delta}$ vary equidistantly in the whole range. The bold trajectory is the model reference trajectory. Figure 4 shows the whole family of time responses in the case of the optimization design where $\boldsymbol{k}_d = [-1.0714;\ -2.6312]$ and the bold trajectory is the model reference trajectory. The comparison of both approaches shows that our method results in a solution which is close to the optimal. The algebraic approach can be very useful and has some advantages especially when we are dealing with a bigger number of parameters or when the problem is non-convex.

6 Conclusion

In this paper we have shown the new procedure of nearly optimal attitude control based on closest radius solution for the systems with uncertain time-invariant physical parameters. The method is based on minimal distance radius solution of the interval matrix inequality. The proposed algorithm was shown for the pitch angle control of unmanned air vehicle where some of the parameters are uncertain due to the weather conditions and height of the flight. It is shown that using the described analytical design the resulting closed-loop responses are qualitatively very similar as the results obtained by optimization. The proposed algorithm gives a framework to design a control for a wide range of different linear time-invariant processes with uncertain parameters and can be implemented also in the case of non-convex problems. The algorithm gives the analytical way of finding the nearly optimal solution of model reference trajectory tracking in the case of general time-invariant systems with uncertain parameters and can be used when optimization method fails due to the complexity of the problem.

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